STABILITY OF A VISCOELASTIC FILM FLOWING OVER A SUBSTRATE WITH SINUSOIDAL CORRUGATIONS

D. Pettas[1], G. Karapetsas[2,*], Y. Dimakopoulos[1], J. Tsamopoulos[1]
[1]Department of Chemical Engineering, University of Patras, Rio, Greece
[2]Department of Chemical Engineering, Aristotle University of Thessaloniki, Thessaloniki, Greece
(*gkarapetsas@cheng.auth.gr)

ABSTRACT
Viscoelastic film flows driven by a body force can be encountered in various engineering applications which range from coating applications in microelectronics to biomedical flows and biofilms. In the literature elastic phenomena are often overlooked since most theoretical works consider the case of Newtonian liquids. The effects of fluid elasticity, though, can play an important role in applications where a viscoelastic liquid is involved as in the case of a polymeric coating solution. In this study, we perform a linear stability analysis for a liquid, that follows the PTT constitutive equation, flowing over a substrate with sinusoidal corrugations. We develop a 2D finite element model and employ Floquet theory to predict the stability of periodic disturbances of arbitrary wavelengths over deep substrate structures. We will present detailed flow stability maps over a wide range of parameters and discuss about the mechanisms through which elasticity affects the present system. We will also discuss about the stability of the flow when it is subjected to 3D disturbances.

INTRODUCTION
Film flows have drawn considerable attention over the years, mainly due to their inherent complexity and their central importance in many industrial processes with scales ranging from micro to macro. Some prominent examples include spin coating, gravure printing, heat exchangers and adsorption or distillation columns using structured packings, mudslides, and lava flows. In many of these applications, the substrates have some kind of structure, and the flow is typically driven by a body force (such as gravity or centrifugal force) or by the motion of the substrate. A typical characteristic of film flows is the appearance of wavy interfacial instabilities which under conditions can be enhanced or mitigated by the presence of the substrate structure. Another important factor is the rheology of the fluids involved depending on the particular application. For example, coating liquids are typically polymeric solutions which exhibit viscoelastic properties, although the role of viscoelastic effects in the stability of the film flow is not well understood. The goal of the present study is to investigate the effect of viscoelasticity on the stability of films flowing over an undulated topography analyzing in depth the interplay of elasticity along with the flow inertia and capillarity on the stability of films.

A large amount of work by several research groups has been devoted to the study of the film flow over structured surfaces either through experimental[1,2] or theoretical studies[3-9]. Kalliadasis and Homsy[10] have shown that the flow over shallow rectangular topographies is stable for a wide range of the relevant parameters, while these results were also supported by the theoretical work Bielarz and Kalliadasis[11] and experimental works[1,12]. More recently, it has been established that in the case of shallow films flowing along deep sinusoidal corrugations might lead to stabilization of the film flow and/or to unstable isles in the linear stability maps[13-16], while under conditions a short-wave mode may also arise in flows over deep periodic corrugations[17].

All the aforementioned studies considered only the case of Newtonian liquid films. However, in many applications the flowing material is often a polymeric solution or a suspension of particles which exhibit viscoelastic properties. Previous theoretical works have shown that the presence of elasticity may considerably affect the steady flow introducing interesting phenomena on the flow arrangement and the film shape[18-20]. Up to now, the examination of the stability of viscoelastic films has been restricted to flows over flat inclined solid surfaces[21-24]. The goal of the present study is to examine the stability of viscoelastic films flowing over surfaces with sinusoidal corrugations of arbitrary depth. To this end, we will consider a viscoelastic liquid that follows the ePTT constitutive law which allows a realistic variation of the shear and extensional fluid viscosities with the local rate of strain components as encountered in typical polymeric solutions. We will solve the 2D momentum balance and constitutive equations for the velocities and viscoelastic stresses without making any restricting assumptions and will perform a linear stability analysis, employing the Floquet-Bloch theory and assuming that the steady solution is subjected both to 2D and 3D disturbances of an arbitrary wavelength.

PROBLEM FORMULATION

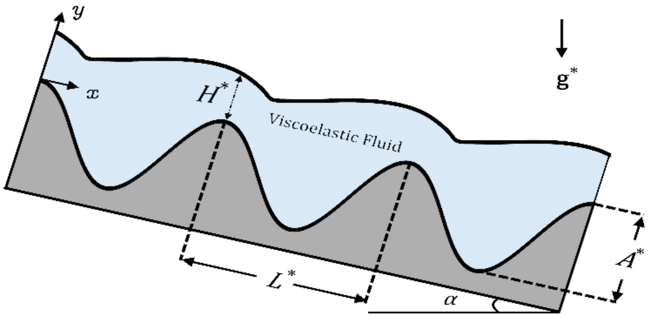

Figure 1 *Cross section of the film flowing over a sinusoidal substrate inclined with respect to the horizontal by an angle α.L^* and A^* are the length and the depth of the unit cell of the substrate, respectively. H^* is the film height at the inlet of the periodic unit cell.*

We consider the free-surface flow of a viscoelastic liquid film driven by gravity along an inclined sinusoidally corrugated substrate normal to the main flow direction, see Fig. 1. In what follows, the superscript *"*"* will indicate a dimensional quantity. The function that describes the shape of the wall is given by the expression:

$$f^*_{Wall}(x^*) = \frac{A^*}{2}\left(\cos\left(\frac{2\pi x^*}{L^*}\right) - 1\right) \tag{1}$$

where L^* and $A^*$are the dimensional length and depth of the unit cell. The liquid is considered to be incompressible, with constant density ρ^*, surface tension σ^*, relaxation time λ_e^* and total zero-shear viscosity $\mu^* = \mu_p^* + \mu_s^*$, where μ_s^* and μ_p^* are the viscosities of the solvent and the polymer, respectively. The primitive flow input is the volumetric flow rate per unit length normal to the film cross-section, q^*. The flow is described using a Cartesian coordinate system with its origin located at the entrance of the flow domain, with the *x*-axis and y-axis in the direction parallel and normal to the wall at *x=0*, respectively (see Fig. 1).

The flow is governed by the momentum and mass conservation equations, which under the Arbitrary Eulerian-Langrangian (ALE) formulation in dimensional form are given by:

$$\rho^*(\partial \boldsymbol{u}^*/\partial t^* + (\boldsymbol{u}^* - \boldsymbol{u}_m^*)\cdot\nabla\boldsymbol{u}^*) + \nabla P^* - \nabla\cdot\boldsymbol{\tau}^* - \rho^*\boldsymbol{g}^* = \boldsymbol{0}\,,\ \nabla\cdot\boldsymbol{u}^* = \boldsymbol{0} \tag{2}$$

where $\boldsymbol{u}^* = \left(u_x^*, u_y^*, u_z^*\right)^T$, P^*, $\boldsymbol{\tau}^*$, denote the velocity, pressure and stress fields, respectively, and $\boldsymbol{u}_m^* = \frac{\partial \boldsymbol{x}^*}{\partial t^*}$ the velocity of the mesh nodes in the flow domain. We also define the unit gravity vector as $\boldsymbol{g}^* = g^*(\sin a, -\cos a)^T$.

The extra stress tensor, $\boldsymbol{\tau}^* = 2\mu_s^*\dot{\boldsymbol{\gamma}}^* + \boldsymbol{\tau}_p^*$, is split into a purely Newtonian part $2\mu_s^*\dot{\gamma}^*$, where $\dot{\gamma}^* = \frac{1}{2}\left(\nabla\boldsymbol{u}^* + \nabla\boldsymbol{u}^{*T}\right)$ is the rate of strain, and a polymeric contribution $\boldsymbol{\tau}_p^*$.

To account for the viscoelasticity of the material we use the exponential Phan-Thien Tanner model:

$$\exp\left(\frac{\lambda_e^*\varepsilon}{\mu_p^*}\ \mathrm{trace}(\boldsymbol{\tau}_p^*)\right)\boldsymbol{\tau}_p^* + \lambda_e^*\ \overset{\nabla}{\boldsymbol{\tau}_p^*} = 2\mu_p^*\dot{\boldsymbol{\gamma}}^*, \tag{3}$$

where $\overset{\nabla}{\boldsymbol{\tau}_p^*} = \frac{\partial \boldsymbol{\tau}_p^*}{\partial t^*} + (\boldsymbol{u}^* - \boldsymbol{u}_{\boldsymbol{m}}^*)\cdot\nabla\boldsymbol{\tau}_p^* - \boldsymbol{\tau}_p^*\cdot\nabla\boldsymbol{u}^* - \left(\boldsymbol{\tau}_p^*\cdot\nabla\boldsymbol{u}^*\right)^T$ denotes the upper convective derivative. Clearly, the ePTT model reduces to the Oldroyd-B model by setting $\varepsilon = 0$.

Along the air-liquid interface we apply the following interfacial stress balance:

$$\boldsymbol{n}\cdot(-P^*\boldsymbol{I} + \boldsymbol{\tau}^*) = -\gamma^*\nabla_s\cdot\boldsymbol{n}, \tag{4}$$

where $\boldsymbol{n}$ is the outward unit normal vector to the free surface and γ^* is the surface tension. We also impose the kinematic condition:

$$\boldsymbol{n}\cdot(\boldsymbol{u}^* - \partial\boldsymbol{x}^*/\partial t^*) = 0, \tag{5}$$

while along the walls of the substrate, we impose the usual no-slip, no-penetration boundary conditions. Additionally, we impose periodic boundary conditions in the velocity and stress field between the inflow and the outflow of the domain, assuming the steady flow has the same periodicity as the substrate structure (i.e., we assume that the steady solution is *L*-periodic)

$$\boldsymbol{u}^*|_{x=0} = \boldsymbol{u}^*|_{x=L}, \quad \boldsymbol{n}\cdot\left(-P^*\boldsymbol{I} + \boldsymbol{\tau}^*\right)\big|_{x=0} = \boldsymbol{n}\cdot\left(-P^*\boldsymbol{I} + \boldsymbol{\tau}^*\right)\big|_{x=L} \tag{6}$$

Finally, the film height at the entrance of the unit cell H^* , is determined by requiring that the flow rate is constant.

$$q^* = \int_0^{H^*} u_x^* dy^* \tag{7}$$

As mentioned above, the base flow is steady, two-dimensional and is assumed to be *L*-periodic. We consider the stability of this steady flow subjected to infinitesimal *2D* and *3D* perturbations. To this end, we map the perturbed physical domain (x, y, z) to a known reference domain (η, ξ, ζ). The variables are written in the computational domain and are decomposed into a part which corresponds to the base state solution and an infinitesimal disturbance using the following ansatz:

$$\begin{bmatrix}\boldsymbol{u}\\ P\\ \boldsymbol{\tau}_p\\ x\\ y\\ z\end{bmatrix}(\eta,\xi,\zeta,t) = \begin{bmatrix}\boldsymbol{u}_b\\ P_b\\ \boldsymbol{\tau}_{p,b}\\ x_b\\ y_b\\ z_b\end{bmatrix}(\eta,\xi) + \delta\begin{bmatrix}\boldsymbol{u}'_d\\ P'_d\\ \boldsymbol{\tau}'_{p,d}\\ x'_d\\ y'_d\\ 0\end{bmatrix}(\eta,\xi)e^{\lambda t + i\,k\zeta}, \tag{8}$$

The first terms on the right-hand side of these equations represent the base solution, indicated by the subscript "b", while the second ones are the perturbation, indicated by the subscript "d" while $\delta \ll 1$. Introducing eq. (8) in the weak form of the governing equations we derive a linearized set of equations for the flow in the bulk and the corresponding boundary conditions. According to our ansatz, an exponential dependence on time is assumed; here λ denotes the growth rate. If the calculated λ turns out to have a positive real part, the disturbance grows with time, and therefore the corresponding steady state is unstable. The disturbances $\boldsymbol{u}'_d$, P'_d, $\boldsymbol{\tau}'_{p,d}$, $\boldsymbol{x}'_d$ are discretized employing finite element basis functions in the streamwise and spanwise directions while Fourier modes are employed in the transverse ζ-direction; k denotes the wavenumber of the perturbation in that direction.

For flows over periodic structured surfaces, the most unstable disturbance for the specific system may have a wavelength that exceeds the period of the domain. Thus, the most appropriate and efficient way to deal with this issue is to employ the Floquet-Bloch theory, which allows us to model the flow over a structured surface by considering the small periodic domain of the topography. According to Bloch's theorem, it is sufficient to look for solutions such that the disturbances between the inflow and outflow of the unity cell are related to each other with the following expression

$$\left.\begin{bmatrix} \boldsymbol{u}'_d \\ P'_d \\ \boldsymbol{\tau}'_{p,d} \\ \boldsymbol{x}'_d \end{bmatrix}\right|_{x=L} = \left.\begin{bmatrix} \boldsymbol{u}'_d \\ P'_d \\ \boldsymbol{\tau}'_{p,d} \\ \boldsymbol{x}'_d \end{bmatrix}\right|_{x=0} e^{2\pi Q i} \tag{9}$$

Using this formulation, the unknown disturbances, $(\boldsymbol{u}'_d, P'_d, \boldsymbol{\tau}'_d, y'_d)^T$, will be determined by imposing eq. (9) at the edges of the periodic domain, which enforces that for finite real values of Q the disturbances will not be L-periodic. Disturbances with $Q = 0$ should be distinguished since in that case eq. (9) reduces to typical periodic boundary conditions, and thus this case corresponds to disturbances that have the same period or aliquots of the basic solution, i.e. correspond to super-harmonic instabilities.

RESULTS AND DISCUSSION

We begin our discussion by presenting in Fig. 2 the base state profiles of a viscoelastic liquid film, along with the spatial variation of the xx-component of the polymeric stress tensor, for two different values of the Weissenberg number, $Wi = \lambda_e^* q^* / {H_N^*}^2$ which correspond to liquids with different amounts of elasticity; H_N^* denotes the mean Nusselt film height, defined as $H_N^* = (3\mu^* q^* / \rho^* g^* \sin\alpha)^{1/3}$. As shown, the interplay between inertial forces which push the fluid against the downstream wall and the elastic rebound from the wall are responsible for intensification of the film deformation and the formation of the cusp, see Fig. 2c. In Fig. 2a we also examine the relative amplitude of the free surface, A_{rel}, which is the ratio of the amplitude of the free surface deformation to the amplitude of the substrate as a function of $Re = \mu^* q^* / \rho^*$.

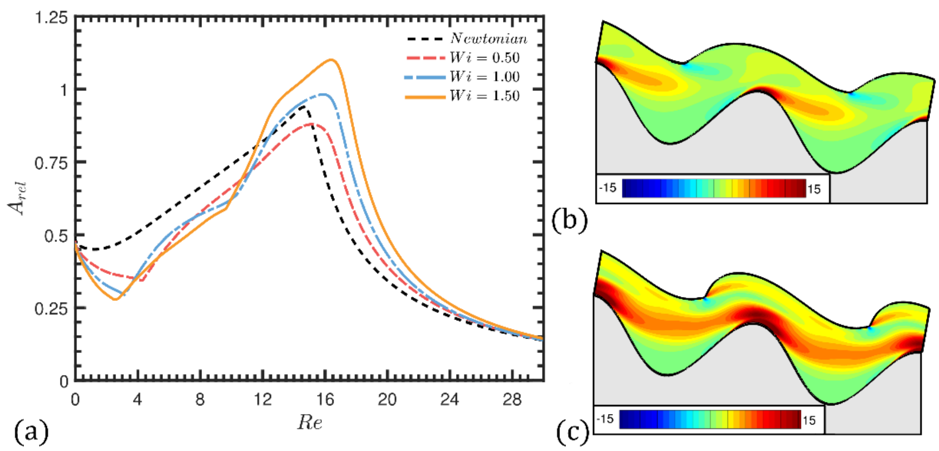

Figure 2. *(a) Relative amplitude of the free surface, A_{rel}, as a function of the Reynolds number for various values of Wi using the Oldroyd-B model ($\varepsilon = 0$). (b,c) Spatial variation of the normal stress component ($\tau_{p,xx}$) for $Wi = 0.5$, and $Wi = 1.5$ calculated at $Re = 15$.*

At intermediate values of the Reynolds number, $2 < Re < 15$, the presence of the substrate structure and the competition between the inertia forces, gravity and surface tension gives rise to an amplification of the steady free surface deformation. The reflected wave along with the surface wave generates a standing wave which becomes amplified in this regime of the Reylonds number and is also observed in the case of Newtonian liquids[12]. The point of resonance arises for a particular value of Re for which the surface velocity of the fluid is equal to the phase velocity of the capillary waves.

Next, we examine the stability of the steady flow when subjected to 2D disturbances. To this end we produce the stability maps shown in Fig. 3 considering values of the Bloch wavenumber Q in the range of [0,0.5] which can be associated with the dimensionless frequency of the disturbances, f. Increasing fluid elasticity, the flow progressively deviates from the Newtonian case[16] as can be seen in Fig. 3a-d for $Wi = 0.5$, 0.75, 1 and 1.5, respectively. For $Wi = 0.5$, the stability map differs in two ways from that of a Newtonian liquid, i.e. two isles of stability arise instead of one for a Newtonian liquid and secondly for low values of elasticity the flow appears to become more stable since the first critical Re increases to 6.89 (for $f = 0.046$ which corresponds to $Q = 0.324$). Further increase of Wi indicates that the bulk fluid elasticity has an overall stabilizing effect on the fluid flow, and the most unstable state is now encountered for longwave disturbances ($f \to 0$). Therefore, we deduce that the elasticity is responsible for dampening high frequency interfacial perturbations.

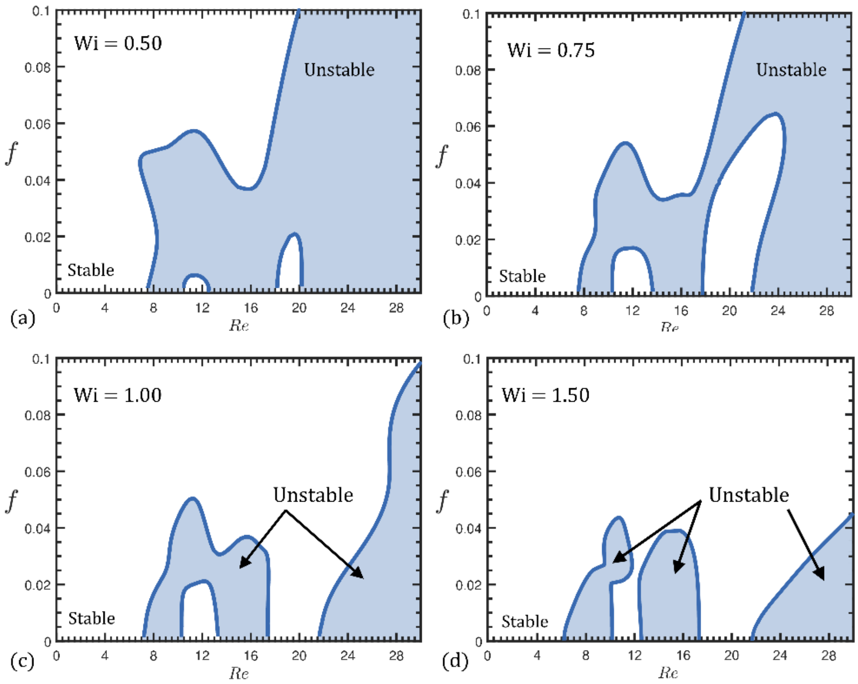

Figure 3. *Effect of the Wi number in the stability diagrams using the Oldroyd-B model. (a)* $Wi = 0.5$*, (b)*$Wi = 0.75$*, (c)* $Wi = 1.0$*, (d)* $Wi = 1.5$

CONCLUSIONS

We carried out a theoretical analysis of the linear stability of a viscoelastic liquid film flowing down an inclined sinusoidal surface. The analysis for the steady state flow revealed that the competition of the inertial forces with the fluid elasticity generates a static hump at the free surface which may be preceded by a cusp. This tends to increase the relative amplitude of the free surface deformation with respect to substrate amplitude. The linear stability, considering infinitesimal perturbations around this base state, predicts a robust stabilization of the fluid flow due to the presence of fluid elasticity. In particular, the spatial variation of polymeric stresses of the base flow create a force that opposes inertia and tends to damp the disturbances for all frequencies; the damping increases with increasing Wi. Interestingly, for large values of Wi, the critical value of Re was calculated to be 4 times larger compared with the case of the Newtonian liquids, while the mechanism of instability is found to be related with the convection of the perturbation of the polymeric stress field by the base state fluid flow. On the contrary, the influence of shear-thinning mechanism destabilizes the flow, by increasing the effective inertia particularly around the maxima of the topography.

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